NOTATION

r, radial distance from well axis; r_c, r_w, radii of column and well; r_m, radius at which melting phase transition occurs; h, m, thickness and porosity of stratum; ℓ, thickness of ice mass; t, time; λ_i and α_i, λ and α, thermal conductivities and diffusivities of ice and porous stratum; ρ_i, L, density and heat of fusion of ice; T_0, T_i, T_m, temperature of earth's surface, porous stratum, and water bubble; $q_1(t)$ and $q(\tau)$, intensity of fictitious heat source, dimensioned and dimensionless; τ, η, x, dimensionless time, thawing radius, and thickness of ice mass; K, M, N, β, α, dimensionless parameters; u, σ, integration variables; δ, length of water bubble; $x_m(t)$, $x_s(t)$, coordinates limiting bubble.

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FINITE-DIFFERENCE SOLUTION OF THE OPTIMIZATION PROBLEM IN HIGH-SPEED HEATING OF A BODY OF SIMPLE SHAPE BY INTERNAL HEAT SOURCES

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A method is proposed for construction of optimal fast-response control of body heating under constraints on the control (internal heat sources) and the temperature field or stress–strain parameters.

Body heating by internal heat sources occurs in modern technological processes, for instance, the induction heating of articles by high-frequency currents [1], in heat exchanger elements [2], in chemical and nuclear reactions [3], etc. Among analogous processes can also be the heating of thin-walled elements during convective heat transfer since in this case the temperature of the external medium is in the right side of the heat-conduction equations [4].

The optimization of body heating relative to fast-response is of direct practical interest to raise the productivity of heater plants [5]. In connection with the limited power of the installation, here, as well as taking into account the requirement of material strength and possibilities of intensive fusion, oxidation, phase microstructure transformation and other phenomena that take place at high temperatures in metals and many materials, constraints are imposed on the control actions, the thermal process parameters, and the stress–strain state [6].

Let us consider the problem of constructing an optimal fast-response control of the heating of homogeneous or inhomogeneous plates, hollow cylinders and spheres by internal heat sources under constraints on the control, the body temperature, the temperature drop, and the thermoelastic stresses in the body.

Heating in the above-mentioned bodies is described by the following boundary-value problems:

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$$a(x)\frac{\partial T}{\partial t}=\frac{1}{x^{\nu}}\frac{\partial}{\partial x}\left(x^{\nu}\lambda(x)\frac{\partial T}{\partial x}\right)+f(x,\ t);\quad (x,\ t)\in V=]k,\ 1[\times]0,\ t'[; \tag{1}$$

$$T(x,\ 0)=\varphi(x),\quad x\in[k,\ 1]; \tag{2}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=k}=H_1[T(k,\ t)-q_1(t)];\qquad -\left.\frac{\partial T}{\partial x}\right|_{x=1}=H_2[T(1,\ t)-q_2(t)], \tag{3}$$

where $\underline{a}(x)$ is a continuous and $\lambda(x)$ a continuously differentiable function.

Let constraints on the control function f (internal heat source intensity) be satisfied:

$$f(x,\ t)\leqslant u(x,\ t) \tag{4}$$

as well as on the temperature field parameters or the stress—strain state

$$FT(x,\ t)\leqslant W(x,\ t),\quad (x,\ t)\in\bar{V}. \tag{5}$$

Here u(x, t) is a given function, F is a linear operator governing the parameters being constained, in particular:

a) body temperature

$$FT(x,\ t)=T(x,\ t); \tag{6}$$

b) temperature drop in the body

$$FT(x,\ t)=T(x,\ t)-\min_{x\in[k,\ 1]}T(x,\ t); \tag{7}$$

c) thermoelastic stresses

$$FT(x,\ t)=\max(\sigma_{xx},\ \sigma_{yy},\ \sigma_{zz}). \tag{8}$$

The following optimization problem is examined. Determine that control function f(x, t) that in a minimal time under the constraints (4) and (5) will carry a body over from the initial state (2) into the final state characterized by a given mean-integrated temperature

$$\frac{\nu+1}{1-k^{\nu+1}}\int_k^1 x^{\nu}T(x,\ t')\,dx=T^*. \tag{9}$$

The modifications presented above for F possess the following property [7]: if $(x_*,\ t_*)$ is the point of the positive maximum of T(x, t) in the domain $\bar{V}$, then $FT(x,\ t)|_{(x_*,t_*)}>0$.

Then on the basis of the theorem in [8] the following can be asserted: if there exists a function that equals the ultimately possible value of the internal heat source intensity or assures ultimately admissible values of the parameters being constrained at each point of the body, then such a function will be optimal control in fast-response for the process under consideration. Analytically this means that at each point of the domain $\bar{V}$ at least one of the equalities

$$f(x,\ t)=u(x,\ t),\quad x\in V_u; \tag{10}$$

$$FT(x,\ t)=W(x,\ t),\quad x\in V_W, \tag{11}$$

should be satisfied, where $V_u\cup V_W=\bar{V}$.

The inequalities (4) and (5) should be satisfied simultaneously with the equalities (10) and (11) in the domain $\bar{V}$. Conditions (4), (5), (10), (11) are optimality conditions for the problem formulated.

Since the domains V_u and V_W are unknown in advance and depend on the thermal state of the body, then the optimization problem is substantially nonlinear. Numerical methods are used for its solutions.

The algorithm to construct the desired control function and the appropriate temperature mode is based on a finite-difference approximation of the boundary-value problem (1)-(3) and the equalities (10), (11).

We partition the domain V by a uniform mesh with step h in the coordinate x and step τ on the time axis. We use the notation $T_i^j = T(x_i, t_j)$; $q_m^j = q_m(t_j)$; $f_i^j = f(x_i, t_j)$, where $x_i = k + ih$; $t_j = j\tau$; $i = \overline{0, n}$; $j = \overline{1, \ell}$; $m = 1, 2$.

The finite-difference analog of the boundary-value problem is a system of linear algebraic equations in T_i^j:

$$a_0T_0^j - c_0T_1^j = d_0T_0^{j-1} + r_1q_1^j + g_0f_0^j,$$
$$-a_iT_{i-1}^j + c_iT_i^j - b_iT_{i+1}^j = d_iT_i^{j-1} + g_if_i^j, \quad 1 \leqslant i \leqslant n-1, \tag{12}$$
$$-a_nT_{n-1}^j + c_nT_n^j = d_nT_n^{j-1} + r_2q_2^j + g_nf_n^j.$$

A purely implicit scheme is used here to construct the difference equations; a_i, b_i, c_i, d_i, r_1, r_2, g, $i = \overline{0, n}$, are the coefficients of the approximation of the boundary-value problem (1)-(3). Formulas for their calculation are presented in [9].

We also approximate the optimality condition in an appropriate manner. The numerical analog of equalities (10) and (11) here has the form

$$f_i^j = u(x_i, t_j); \tag{13}$$

$$\Phi_i^j = W(x_i, t_j), \tag{14}$$

where $i = \overline{0, n}$; $j = \overline{1, l}$; $\Phi_i^j = FT(x, t)|_{(x_i, t_j)}$.

The matrix of the system (12) has a three-diagonal shape with diagonal predominance, which assures correctness and stability of the algorithm by monotonic factorization [10]. The value of the temperature T_i^j, $i = \overline{0, n}$; $j = \overline{1, \ell}$ can be found by the factorization method if the values of the control f_i^j, $i = \overline{0, n}$; $j = \overline{1, \ell}$ and the temperature T_i^{j-1}, $i = \overline{0, n}$; $j = \overline{1, \ell}$ are known. Here $T_i^0 = \varphi(x_i)$, $i = \overline{0, n}$. Since the constraints (4) and (5) should be satisfied at the initial time, then the following inequality is valid

$$F\varphi(x) \leqslant W(0, x), \quad x \in [k, 1].$$

Consequently, it can be assumed that $f_i^1 = u(x_i, \tau)$, $i = \overline{0, n}$.

At the time t_L let the inequality

$$\Phi_N^L > W(x_N, t_L) \tag{15}$$

be satisfied at the point x_N, which contradicts condition (5).

Then to satisfy the optimality conditions (4), (5), (10), (11) it is necessary to require compliance with the equalities

$$f_i^L = u(x_i, t_L), \quad i \neq N; \tag{16}$$

$$\Phi_N^L = W(x_N, t_L). \tag{17}$$

In this case the solution of the system of equations (12) in which $f_i^L = u(x_i, t_L)$ for $i \neq N$ must be found to determine T_i^L, $i = \overline{0, n}$ and the N-th equation must be replaced by (17).

According to the known values of T_i^L, $i = \overline{0, n}$ and from the N-th equation of the original system (12), the value of f_N^L is determined uniquely, which is already not generally equal to the ultimately possible. Thus, for N = 0

$$f_0^L = (a_0T_0^L - c_0T_1^L - d_0T_0^{L-1} - r_1q_1^L)/g_0; \tag{18}$$

for $N = \overline{1, n-1}$

$$f_N^L = (c_NT_N^L - a_NT_{N-1}^L - d_NT_N^{L-1} - b_NT_{N+1}^L)/g_N; \tag{19}$$

for N = n

$$f_n^L = (c_nT_n^L - a_nT_{n-1}^L - d_nT_n^{L-1} - r_2q_2^L)/g_n. \tag{20}$$

Since the desired optimal control should satisfy condition (4), then at the time t_L the inequality

$$f_N^L \leqslant u(x_N,\ t_L) \tag{21}$$

should be satisfied, which indicates the necessity for lowering the heat source power at the time the parameter being constrained reaches the ultimately allowable value. This condition is satisfied for the majority of practical problems.

If condition (21) is not satisfied, then the method being proposed does not permit construction of the function satisfying (4), (5), (10), and (11).

Let inequality (21) be satisfied, and at a certain time $t_j > t_L$ let the desired control f_N^j exceed the ultimately possible value $u(x_n,\ t_j)$. Then again we should set $f_N^j = u(x_N,\ t_j)$, and the values of $T_i^j,\ i=\overline{0,\ n}$, will be determined from the system (12). If inequality (15) is satisfied at the moment of time t_L at several points $x_{N_1},\ x_{N_2},\ \ldots,\ x_{N_s},\ s \leqslant n$, then the value of $T_i^L,\ i=\overline{0,\ n}$, is determined from the system $N_1, N_2, \ldots, N_s$, whose equations are replaced by equations of the form (17). The unknown values of the control functions are determined from (18)-(20). The algorithm to determine f_i^j and T_i^j, $i = 0,\ n;\ j = 1,\ l$ at different stages of the heating is thereby given.

Let us use the algorithm described to construct an optimal control for specifically given constraints.

For a constraint on the body temperature $\Phi_i^j = T_i^j$. In the more general case the system of equations to determine T_i^j will contain $s \leq n$ equations

$$T_{N_m}^j = W(x_{N_m},\ t_j),\quad m=\overline{1,\ s},$$

and the rest will agree with the equations of system (12).

It is easy to see that this is a tridiagonal system, has diagonal predominance, and therefore, can easily be solved by the factorization method.

The finite-difference approximation of the constraint b) will have the form

$$\Phi_i^j = T_i^j - T_r^j. \tag{22}$$

The point x_r satisfies the condition $T_r^j = \min_i \{T_i^j\},\ i=\overline{0,\ n}$. In this case the system of linear equations in T_i^j will contain $s \leq n$ equations

$$T_{N_m}^j - T_r^j = W(x_{N_m},\ t_j),\quad m=\overline{1,\ s},$$

and the rest will agree with equations of the system (12).

The matrix of the system constructed possesses the property of diagonal predominance and its structure is such that it permits stable determination of the solution by the method of factorization for complex systems [10].

If it turns out during further calculations that $T_p^j = \min_i \{T_i^j\},\ i=\overline{0,\ n};\ p \neq r$, then the computations must be repeated by replacing T_r^j by T_p^j in (22).

In the case c), for instance, for constraints on the compressive stress in a homogeneous plate [6]

$$FT(x,\ t) = T(x,\ t) - \int_0^1 T(x,\ t)\, dx$$

Φ_i^j will have the form

$$\Phi_i^j = \sum_{p=0}^{n} (\delta_{pi} - \alpha_p)\, T_p^j,$$

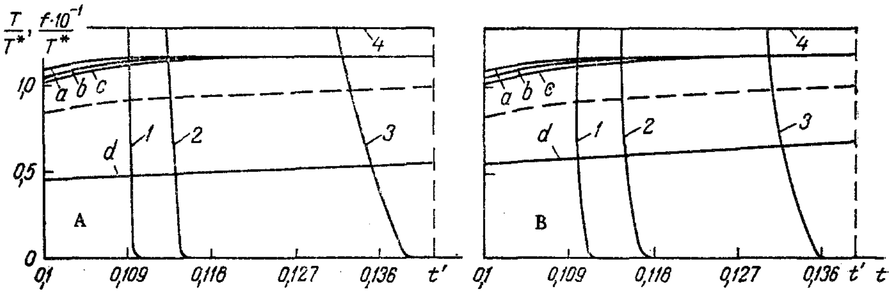

Fig. 1. Optimal control of heating a hollow cylinder (A, H_2 = 6) and a hollow sphere (B, H_2 = 4) for a constraint on the temperature (W(x, t) = 1.2; u(x, t) = 13, x_1 = 0.5; x_2 = 0.575; x_3 = 0.65; x_4 = 1).

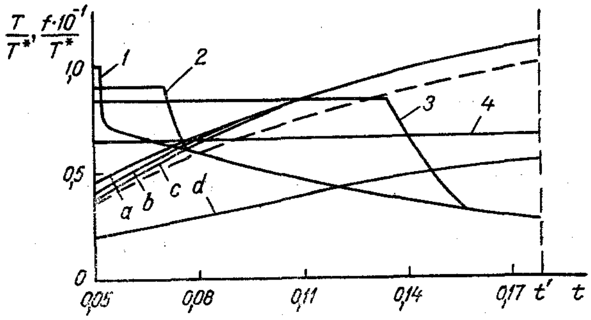

Fig. 2. Optimal control of heating a plate under a constraint on the compressive stress (W(x, t) = 0.05; u(x, t) = −3x + 10; H_2 = 4; x_1 = 0; x_2 = 0.3; x_3 = 0.5; x_4 = 1).

where δ_{pi} is the Kronecker delta, and $\alpha_0 = \alpha_n = h/2$; $\alpha_i = h$; $i = \overline{1, n-1}$.

As earlier, in the case of surpassing the constant (5) at the points $x_{N_1}, \ldots, x_{N_s}$ at the time t_j, the system to determine T_i^j, $i = \overline{0, n}$, will contain $s \leq n$ equations

$$\sum_{p=0}^{n} (\delta_{pN_m} - \alpha_p) T_p^i = W(x_{N_m}, t_j), \quad m = \overline{1, s},$$

and the rest will agree with the equations of the system (12).

It can easily be shown that the matrix of this system has diagonal predominance; consequently, the solution can be determined stably by using a certain modification of the factorization method for complex systems.

As an example of realizing the proposed method, an optimal control of the heating of a plate, hollow cylinders, and spheres was computed for $a(x) = \lambda(x) = 1$; $\phi(x) = 0$; $H_1 = 0$, $q_2 = 0$.

The change in the temperature $T(x_i, t)/T^*$, $i = \overline{1, 4}$, is shown in Figs. 1 and 2 by the solid curves a, b, c, d, and the change in the optimal control $f(x_i, t)/T^*$, $i = \overline{1, 4}$, during heating by curves 1-4. The change in the relative mean integrated temperature is shown by dashed lines.

As is seen from the figures, the optimal fast-response control at the point x_4 equals the ultimately allowable in the extent of the whole heating process while the optimal control at the points x_1, x_2, x_3 is in two stages: in the initial heating period it equals the ultimately possible value, and after emergence at the constraint assures the ultimately allowable values of the parameters being constrained at these points.

Remark. This numerical solutions method for the problem of optimal fast-response heating can evidently be applied also to piecewise-homogeneous bodies of simple shape. To do this it is necessary to select an appropriate method of approximating the heat conduction equation and the boundary conditions. The heat flux or the temperature on the body surface can also be given as the latter. As the final target we can have the heating to a given temperature at a certain fixed point.

NOTATION

$x = x_*/\ell$, a dimensionless coordinate; $k = 0$, $\ell = \bar{h}$ in the case of a plate ($\nu = 0$), $k = R_1/R_2$, $\ell = R_2$ in the case of hollow cylinders ($\nu = 1$) and spheres ($\nu = 2$); $\bar{h}$, R_1, R_2, respectively, plate thickness and the inner and outer radii of the cylinder or sphere, m; $a(x) = a_*(x)/a_*(k)$, $a_*(x)$, bulk specific heat, $J/(m^3 \cdot K)$; $\lambda(x) = \lambda_*(x)/\lambda_*(k)$, $\lambda_*(x)$, heat-conduction coefficient, $W/(m \cdot K)$; $t = \lambda_*(k)t_*/a_*(k)\ell^2$, t_*, time, sec; $T(x, t)$, body temperature, °K; $f(x, t) = f^*(x, t)\ell^2/\lambda_*(x)$, $f^*(x, t)$, internal heat source intensity, W/m^3; H_1, H_2, dimensionless heat-transfer coefficients; $q_1(t)$, $q_2(t)$, temperatures of the external medium, °K; t', optimal heating time, sec; σ_{xx}, σ_{yy}, σ_{zz}, dimensionless principal stress tensor components.

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